Asymptotic Results for Configuration Model Random Graphs with Arbitrary Degree Distributions

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Abstract

We consider a model for generating a random graph using the configuration model. In the configuration model, each node draws a degree independently from a marginal degreed distribution and endpoints pair randomly. We establish non-trivial bounds on the expected sizes of "buckets" for large graphs. We define nodes i and j in a graph as neighbors if they share an edge, and we define the "bucket" associated with node i as the set of nodes that are its neighbors and have degree greater than node i. We formalize this argument by providing an analysis for the expected number of items and pairs in a "bucket" for arbitrarily specified degree distributions, which include power laws.

1 Introduction

Many situations can be modeled using networks (or equivalently, graphs). The world wide web, social networks, and genomic aberrations are just a few examples. While the focus in most cases is on networks with only a single edge between two nodes, there are cases where multiple edges can exist between two nodes. Webpages with multiple links between each other are a very common case. Occasionally there may be a node with an edge connecting it to itself (called a self-loop). Webpages are also a common source of self-loops. Go to www.apple.com to see an example of all three different types of links (edges).

Recently there has been an interest in an ability to make sound statistical inferences about features of a network as well as the network itself. The features that are most commonly of interest are the community structure within the network as well as shapes made by the edges, such as triangles. This requires properly modeling a joint distribution for the network as well as the marginal distributions of the features of network. Erdős and Rényi (1959) were the first to provide a joint distribution for a random graph by modeling the edges between each node as independent bernoulli events. However, this model is inappropriate for most applications as most of the degree distribution observed in real world networks appear to be best modeled by a power law distribution. Networks exhibiting power law degree distributions are referred to as scale-free networks [Dorogovtsev and Mendes (2002)]. Consequently there is a need for a more flexible method of modeling both joint and marginal distributions.

Many alternatives to the Erdös Rényi random graph have been presented, most attempting to address the scale-free behavior. Specifically, most real world networks are thought to be best modeled with heavy tailed power law degree distributions having an exponent $2 < \beta < 3$. Barabási and Albert (1999) suggested the "preferential attachment" model and showed that the degree distribution asymptotically follows a power law distribution. Watts and Strogatz (1998) studied the behavior of graphs upon rewiring as the probability of connecting to other nodes in the graph was varied. Under certain conditions a scale free behavior was observed. One of the most flexible mechanisms for generating graphs is the configuration model, as originally studied by Bender and Canfield (1978) and Wormold (1978). The configuration model generates undirected random graphs according to an arbitrary specified degree distribution, including power laws. Edges are randomly paired with every pairing equally likely, including self-loops and multi-edges, so that the resulting graph may not be simple.

The enumeration of triangles in a network, also known as node counting, is an important part of many algorithms analyzing clustering and neighborhoods within complex networks [Latapy (2008)]. The motivation for developing efficient node counting algorithms is the ubiquity of measures such as the clustering coefficient [Albert and Barabási (2002)] and the transitivity ratio, which are considered to be useful descriptors of large networks. Berry et al. (2009) require the the number of triangles to compute edge weights in their work improving the resolution limits (ability to detect both small and large communities) of community detection algorithms. While the application to computing edge weights is just one specific example, it was the node counting algorithm used with this application that provided the motivation for this analysis. The method for listing triangles is by giving each node a "bucket" where each edge in the graph is placed into the bucket of its endpoint of lowest degree. Pairs of edges in each bucket are then checked for connecting edges which complete a triangle. Cohen (2009) developed this algorithm and showed that it enumerates the number of triangles in a graph.

To understand the performance of Cohen's method for listing triangles, (which impacts the computational efficacy of community detection algorithms for example) it becomes of interest to quantify both the size and number of pairs which might be expected of a typical bucket in a large random graph. Using the configuration model as the graph generating mechanism, we show that the expected size of a "bucket" is finite provided the degree distribution has a finite first moment. We further show the expected number of pairs in a bucket to be finite provided the degree distribution has a finite 4/3 moment. That is, Cohen's triangle listing algorithm will scale linearly provided the degree distribution has a finite 4/3 moment.

2 The Results

In this section, a class of random graphs is described and, for these, a non-trivial bound on the expected sizes of "buckets" in large graphs is established. A non-trivial bound on the expected number of pairs of nodes in a bucket is established as well under mild conditions. Random graphs are generated using the configuration model which allows for the degree distributions to be quite *arbitrarily* specified and include power laws in particular.

Consider a graph with n nodes, where the degree of the ith node is D_i, $i = 1, \ldots, n$. Suppose the collection of degrees $D_1, \ldots, D_n$ are independent and identically distributed. The common distribution, which is described in terms of D_1 in the following (without loss of generality), follows a discrete probability mass function

$$f(d) = \mathrm{P}(D_1 = d), \quad \text{integer } d \in [\ell, \infty), \tag{1}$$

where ℓ is a nonnegative integer (a parameter) which may be specified in the model (e.g., $\ell = 1$). A degree distribution must be specified subject to the usual probability constraint $\sum_{d=\ell}^{\infty} f(d) = 1$. To obtain a bounded degree distribution, assume $f(d) = 0$, $d > u$ for some selected upper bound $u \geq \ell$.

After generating degrees $D_1, \ldots, D_n$ for the nodes, the system is in a state where the ith node has D_i stubs or half-edges. Assuming $S_n = \sum_{i=1}^{n} D_i$ is even, the S_n stubs are then randomly paired, with every "pairing configuration" assumed to be equally likely. If S_n is odd, we randomly pick one integer I from $\{1, \ldots, n\}$, replace D_I with $D_I + 1$, and perform pairing. For odd S_n, this additional randomization step has little influence on overall probability structure of individual nodes in large graphs ($n \to \infty$). This graph generation does allow multigraphs in that self-loops are possible as well as multiple edges with neighboring nodes.

Nodes i and j are defined to be neighbors if they share at least one edge in the graph. Fix an arbitrary node i among the n nodes of the graph and define a "bucket"

$$\mathcal{B}_{i,n} = \{j : i \neq j, D_i \leq D_j, \text{node } i \text{ and node } j \text{ are neighbors}\},$$

which corresponds to the set of all neighbors of node i having degree at least as great as that of node i. Let $N_{i,n} = |\mathcal{B}_{i,n}|$ be the size of the bucket for node i in a size n graph. The number of possible node pairs that can be formed from nodes in the bucket $\mathcal{B}_{i,n}$ is

$$\binom{N_{i,n}}{2} = \frac{N_{i,n}(N_{i,n} - 1)}{2}.$$

To show the "bucket" algorithm is an effective node counting algorithm, it is necessary to evaluate the expected value $\binom{N_{i,n}}{2}$ as the number of nodes $n \to \infty$. Letting E represent the expectation operator, the expected value is denoted as $\mathrm{E}\binom{N_{i,n}}{2}$. Under a mild moment condition on the degree distribution, an explicit expression for

$$\lim_{n\to\infty} \mathrm{E}\binom{N_{i,n}}{2}$$

is provided and nontrivially shown to be finite. Additionally, the limiting form of expected bucket size $\mathrm{E}N_{i,n}$ is provided.

To state the result, recall that the rth moment, $r > 0$, of the degree distribution is given by $\mathrm{E}D_1^r = \sum_{d=\ell}^{\infty} d^r f(d)$ and, in particular, $\mathrm{E}D_1 = \sum_{d=\ell}^{\infty} d f(d)$ is the expected value or mean of the degree distribution.

Theorem 1 *Under the above mechanism for generating random graphs, suppose $\ell \geq 1$ in the degree distribution (1).*
(i) If $\mathrm{E}D_1 < \infty$, then as $n \to \infty$,

$$\mathrm{E}N_{i,n} \to \frac{1}{\mathrm{E}D_1} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} d_1 d_2 \; f(d_1) f(d_2) < \infty$$

(ii) If $\mathrm{E}D_1^{4/3} < \infty$, then as $n \to \infty$,

$$\mathrm{E}\binom{N_{i,n}}{2} \to \frac{1}{2} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} f(d_1) f(d_2) f(d_3) \frac{d_1 d_2}{\mathrm{E}D_1} \frac{(d_1-1)d_3}{\mathrm{E}D_1} < \infty.$$

To frame the results in Theorem 1, note that the bucket size $N_{i,n}$ of node i (a function of the graph size n) is always bounded by the degree $D_i + 1$ of node i plus one (not depending on n and potentially incremented by 1 in the act of random wiring); that is, $N_{i,n} \leq D_i + 1$. Consequently, it follows that

$$\limsup_{n\to\infty} \mathrm{E}\binom{N_{i,n}}{2} \leq \mathrm{E}\binom{D_i+1}{2} \leq \mathrm{E}D_1^2.$$

Hence, whenever $\mathrm{E}D_1^2 < \infty$, the value of $\mathrm{E}\binom{N_{i,n}}{2}$ will be trivially finite for all graph sizes n; this includes the special case where the degree distribution is bounded ($f(d) = 0$ for $d > u$, some u). A nontrivial aspect of Theorem 1(ii) that the limit of $\mathrm{E}\binom{N_{i,n}}{2}$ can also finitely exist even in cases where the degree distribution is so heavy in its tail probabilities that $\mathrm{E}D_1^2 = \infty$ holds and, in this situation, $\mathrm{E}D_1^2$ can no longer provide a trivial finite bound on $\mathrm{E}\binom{N_{i,n}}{2}$. This exact feature holds for degree distributions specified for many power laws of interest, which are defined on $[1, \infty)$ and have probabilities

$$f(d) = L(d) d^{-\beta}, \quad \text{integer } d \geq 1, \tag{2}$$

where $\beta \geq 1$ is the tail index and $L(\cdot)$ is a slowly varying function at ∞ (i.e., for any $t > 0$, $L(td)/L(d) \to 1$ as $d \to \infty$.) For concreteness, the results of Theorem 1 are recast for the special case of power laws (2) under the additional assumption that the slowly varying function is bounded away from zero; this implies that, for $r > 0$, $\mathrm{E}D_1^r$ is finite if and only if $r + 1 < \beta$

and encompasses scenarios such as constant $L(d) = C$ or $L(d) = C\log(d)$. (Without this constraint, $\mathrm{E}D_1^r$ for $r = \beta - 1$ could be finite or not, depending more closely on $L(\cdot)$.)

Corollary 1 *Under the mechanism for generating random graphs, suppose the degree distribution is a power law* (2) *where the slowly varying function satisfies* $\liminf_{d\to\infty} L(d) > 0$. *Then, the following table summaries moments and limits as finite (F) or infinite* (∞)*, where the values of finite limits are as in Theorem 1 (denoted as "F-Th1").*

	$\mathrm{E}D_1$	$\lim_{n\to\infty} \mathrm{E}N_{i,n}$	$\mathrm{E}D_1^{4/3}$	$\lim_{n\to\infty} \mathrm{E}\binom{N_{i,n}}{2}$	$\mathrm{E}D_1^2$
$\beta \le 2$	∞	∞	∞	∞	∞
$\beta \in (2, 2\frac{1}{3}]$	*F*	*F-Th1*	∞	∞	∞
$\beta \in (2\frac{1}{3}, 3]$	*F*	*F-Th1*	*F*	*F-Th1*	∞
$\beta > 3$	*F*	*F-Th1*	*F*	*F-Th1*	*F*

Corollary 1 implies that, for power laws (2), the limiting expectation in Theorem 1(ii) [or (i)] is finite if and only if $\mathrm{E}D_1^{4/3} < \infty$ [$\mathrm{E}D_1 < \infty$] if and only if $\beta > 2\frac{1}{3}$ [$\beta > 2$]. Perhaps surprisingly, the expected pairs from a bucket will remain finite as the graph grows for heavy tailed power laws with index $\beta \in (2\frac{1}{3}, 3]$ for which $\mathrm{E}D_1^2 = \infty$ holds.

3 Proof of Theorem 1

The proof of Theorem 1 (assuming $\ell > 0$) is outlined in a series of steps, which are divided into Sections 3.1-3.5 for clarity.

In a graph with n nodes, fix an arbitrary node i, WLOG say $i = 1$. First, the expressions for the finite-sample expectations $\mathrm{E}N_{1,n}$ and $\mathrm{E}\binom{N_{1,n}}{2}$ are given. To this end, let $B_n(j)$ denote the generic event that a node j is a neighbor of a node 1. If A and B represent two events/sets, let "A, B" denote their set intersection. Using indicator functions ($\mathbb{I}(A) = 1$ if event A holds and otherwise $\mathbb{I}(A) = 0$), write the count $N_{1,n} = \sum_{j=2}^n \mathbb{I}\big(B_n(j), D_j \ge D_1\big)$ so that

$$\mathrm{E}N_{1,n} = \sum_{j=2}^{n} \mathrm{E}\mathbb{I}\big(B_n(j), D_j \ge D_1\big) = (n-1)\mathrm{P}(B_n(2), D_2 \ge D_1), \qquad (3)$$

using $\mathrm{E}\mathbb{I}(B_n(j), D_j \geq D_1) = \mathrm{P}(B_n(j), D_j \geq D_1) = \mathrm{P}(B_n(2), D_2 \geq D_1)$, and similarly

$$\begin{aligned}
\mathrm{E}N_{1,n}^2 &= \mathrm{E}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n} \mathbb{I}(B_n(j), D_j \geq D_1, B_n(k), D_k \geq D_1) \\
&= (n-1)(n-2)\mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1) + (n-1)\mathrm{P}(B_n(2), D_2 \geq D_1)
\end{aligned}$$

using $\mathbb{I}(B_n(j), D_j \geq D_1, B_n(k), D_k \geq D_1) = \mathbb{I}(B_n(j), D_j \geq D_1)$ if $j = k$. Substitution of these expressions yields

$$\mathrm{E}\binom{N_{1,n}}{2} = \frac{1}{2}(\mathrm{E}N_{1,n}^2 - \mathrm{E}N_{1,n}) = \binom{n-1}{2}\mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1). \tag{4}$$

Section 3.1 provides expansions which allow the expected values in (3)-(4) to be expressed as

$$\begin{aligned}
\mathrm{E}N_{1,n} &= (n-1)\Delta_{1,n} + O\left(\frac{1}{n}\right) \qquad (5) \\
\mathrm{E}\binom{N_{1,n}}{2} &= \frac{(n-1)(n-2)}{2}\Delta_{2,n} + O\left(\frac{1}{n}\right),
\end{aligned}$$

where $\Delta_{1,n}, \Delta_{2,n}$ are decompositions of the probabilities $\mathrm{P}(B_n(2), D_2 \geq D_1), \mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1)$ into sums of probabilities of more elemental events. Section 3.1 also provides a further technical result, which helps to show

$$\lim_{n\to\infty}(n-1)\Delta_{1,n} = \frac{1}{\mathrm{E}D_1}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} d_1 d_2\ f(d_1)f(d_2) \tag{6}$$

as established in Section 3.4, as well as

$$\lim_{n\to\infty}\frac{(n-2)(n-1)}{2}\Delta_{2,n} = \frac{1}{2(\mathrm{E}D_1)^2}\sum_{d=\ell}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty} d_1 d_2 (d_1-1) d_3\ f(d_1)f(d_2)f(d_3) < \infty, \tag{7}$$

which is proven in Section 3.5. Theorem 1 then follows from (5)-(7).

3.1 Reduction Step and Technical Results

Lemma 1 below gives an expansion of the probabilities stated in (3)-(4) and justifies the reduction step (5). Lemma 2 gives the limiting values of the reductions as well as mild conditions for the limits to exist. To state the result, additional notation is required as well as an explicit function for the probability of an edge forming between two nodes with the configuration model as the graph generating mechanism. Write $f_{S_{n-k}}(s) = \mathrm{P}(S_{n-k} = s)$ to denote the probability function of the partial sum $S_{n-k} = \sum_{i=k+1}^{n} D_i$; in a graph with n nodes, S_{n-k} is the sum of degrees for all nodes excluding nodes $1, \dots, k$.

Let A_1 and A_2 be two nodes with a_1 and a_2 stubs (two stubs connect to form one edge) respectively in a graph with n nodes. Let s be the sum of the remaining nodes in the graph such that $a_1 + a_2 + s \geq 2$ and $a_1 + a_2 + s$ is even. Since the configuration allows for self-loops and multi-edges, let k be the number of self-loops made by the stubs of node A_1. Then there are

$$\frac{1}{\left(\frac{a_1+a_2+s}{2}\right)!}\binom{a_1+a_2+s}{2}\binom{a_1+a_2+s-2}{2}\cdots\binom{2}{2} = \frac{1}{2^{(a_1+a_2+s)/2}}P^{a_1+a_2+s}_{(a_1+a_2+s)/2}$$

possible configurations in a graph with $a_1 + a_2 + s$ edges. $P^y_x = y!/(y-x)!$ is the permutation function for integers $y \geq x \geq 0$ and $P^y_x = 0$ if $x > y \geq 0$. In order to calculate the probability that node A_1 is not a neighbor of node A_2 given a_1 and a_2 stubs, respectively, in a graph with $a_1 + a_2 + s$ stubs, the number of configurations where all the stubs of node A_1 do not connect to any of the stubs of node A_2 must be computed.

Given k self loops on node A_1, there are $\frac{1}{k!}\binom{a_1}{2}\binom{a_1-2}{2}\cdots\binom{a_1-2(a_1-1)}{2} = \frac{P^{a_1}_{2k}}{k!2^k}$ unique pairings of the stubs to form self-loops. For the a_1-2k remaining stubs, there are $\frac{s!}{[s-(a_1-2k)]!} = P^s_{a_1-2k}$ possible pairings with the s stubs in the graph not belonging to node A_1 or A_2. For the remaining $s-(a_1-2k)+a_2$ stubs, there are $\frac{1}{2^{(s-(a_1-2k)+a_2)/2}}P^{s-(a_1-2k)+a_2}_{(s-(a_1-2k)+a_2)/2}$ possible pairing with stubs not belonging to node A_1. Hence, define a function

$$h(a_1, a_2, s^*, k) = P^{a_1}_{2k}\frac{2^{a_1-2k}}{k!}P^{s^*}_{a_1-2k}\frac{1}{P^{a_1+a_2+s^*}_{(a_1+a_2+s^*)/2}}P^{s^*-(a_1-2k)+a_2}_{[s^*-(a_1-2k)+a_2]/2}, \tag{8}$$

where $s^* = s$ if a_1+a_2+s is even and $s^* = s+1$ if a_1+a_2+s is odd. A similar version of (8) was derived by Wormold (1978) for d-regular random graphs. This function represents the probability of nodes A_1 and A_2 not sharing any edges conditioned on having node A_1 have k self-loops. An unconditioned argument requires summing over the $k = 0, \dots, \lfloor a_1/2 \rfloor$ possible self-loops that can be made by node A_1. This results in

$$p_1(a_1, a_2, s^*) = \sum_{k=0}^{\lfloor a_1/2 \rfloor} h(a_1, a_2, s^*, k). \tag{9}$$

Therefore, the probability that nodes A_1 and A_2 share at least one edge is $1 - p(a_1, a_2, s^*)$. This function will be useful in proving Lemma 2(i).

For the proof of Lemma 2(ii), the probability that node A_1 and node A_2 share at least one edge and node A_1 and node A_3 share at least one edge must be computed. This probability is given by

$$p_2(a_1, a_2, a_3, s) = 1 - p_1(a_1, a_2, a_3 + s) - p_1(a_1, a_3, a_2 + s) + p_1(a_1, a_2 + a_3, s). \tag{10}$$

This follows by the inclusion exclusion principle and the probability function (8). In other words, this is equivalent to the probability node A_1 shares at least one edge with node A_2 plus the probability node A_1 shares at least one edge with node A_3, minus the probability probability node A_1 share at least one edge with nodes A_2 or A_3.

3.2 Lemma 1

Lemma 1 *Under the mechanism for generating random graphs, as $n \to \infty$,*

$$\begin{aligned} \mathrm{P}(B_n(2), D_2 \geq D_1) &= \Delta_{1,n} + O\left(\frac{1}{n^2}\right) \\ \mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1) &= \Delta_{2,n} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

where, for $p_2(d_1, d_2, d_3, s) \equiv 1 - p_1(d_1, d_2, s + d_3) - p_1(d_1, d_3, s + d_2) + p_1(d_1, d_2 + d_3, s)$,

$$\begin{aligned} \Delta_{1,n} &\equiv \sum_{d_1=\max\{\ell,1\}}^{\infty} \sum_{d_2=d_1}^{\infty} f(d_1) f(d_2) \sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s)[1 - p_1(d_1, d_2, s)] \\ \Delta_{2,n} &\equiv \sum_{d_1=\max\{2,\ell\}}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} f(d_1) f(d_2) f(d_3) \sum_{s=(n-3)\ell}^{\infty} f_{S_{n-3}}(s) p_2(d_1, d_2, d_3, s). \end{aligned}$$

Addressing the case $\ell > 0$ so that $\max\{\ell, 1\} = \ell$, the lemma essentially states that complications in the graph construction due to any additional randomization when $D_1 + D_2 + S_{n-2}$ (or $D_1 + D_2 + D_3 + S_{n-3}$) is odd can be ignored. In this case, S_{n-2} (or S_{n-3}) is incremented by 1. This introduces a small order error compared to the main component $\Delta_{1,n}$ in the probability $\mathrm{P}(B_n(2), D_2 \geq D_1)$ (or the main term $\Delta_{2,n}$ in $\mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1)$). Ignoring such additional randomization, $\Delta_{1,n}$ is derived by decomposing $\mathrm{P}(B_n(2), D_2 \geq D_1)$ as the sum of probabilities $\mathrm{P}(B_n(2), D_2 \geq D_1, D_1 = d_1, D_2 = d_2, S_{n-2} = s)$ over all values of $d_1, d_2 \in [\ell, \infty), s \in [(n-2)\ell, \infty)$. Using conditional probabilities, it follows

$$\begin{aligned} & \mathrm{P}(B_n(2), D_2 \geq D_1, D_1 = d_1, D_2 = d_2, S_{n-2} = s) \\ = \; & \mathbb{I}(d_2 \geq d_1) \cdot \mathrm{P}(B_n(2)|D_1 = d_1, D_2 = d_2, S_{n-2} = s) \cdot \mathrm{P}(D_1 = d_1, D_2 = d_2, S_{n-2} = s) \end{aligned}$$

where $\mathrm{P}(D_1 = d_1, D_2 = d_2, S_{n-2} = s) = \mathrm{P}(D_1 = d)\mathrm{P}(D_2 = d_2)\mathrm{P}(S_{n-2} = s)$ by independence of the node degrees. Above $\mathrm{P}(B_n(2)|D_1 = d_1, D_2 = d_2, S_{n-2} = s)$ is the conditional probability that nodes 1 and 2 are neighbors in the case that $D_1 = d_1, D_2 = d_2, S_{n-2} = s$. This is expressed as $\mathrm{P}(B_n(2)|D_1 = d_1, D_2 = d_2, S_{n-2} = s) = 1 - p_1(d_1, d_2, s)$ as in (9). The main component $\Delta_{2,n}$ of $\mathrm{P}(B_n(2), D_2 \geq D_1, B_n(3), D_3 \geq D_1)$ is derived in a similar manner.

Sections 3.4 and 3.5 establish the limits stated in (6) and (7), respectively. These derivations are based on a technical result, given in Lemma 2 below, that allows the Lebesgue Dominated Convergence Theorem (DCT) to be applied under the moment assumptions $\mathrm{E}D_1 < \infty$ or $\mathrm{E}D_1^{4/3} < \infty$ in Theorem 1. Briefly recall the main aspects of the DCT. If μ denotes a generic measure on a measurable space $(\Omega, \mathcal{F})$ (cf. Ch. 2, Athreya and Lahiri, 2006) and if h_n generically denotes a sequence of (measurable) functions, the DCT essentially says that the limit of integrals equals the integral of limits

$$\lim_{n\to\infty} \int_\Omega h_n d\mu = \int_\Omega \lim_{n\to\infty} h_n d\mu,$$

if three components can be verified:

(i) there exists a measurable function h where $\lim_{n\to\infty} h_n = h$ pointwise on Ω (except possibly on a set $A \in \mathcal{F}$ with $\mu(A) = 0$);

(ii) there exists a measurable function g where $|h_n| \leq g$ (except possibly on a set $A_n \in \mathcal{F}$ with $\mu(A_n) = 0$) for each $n \geq 1$;

(iii) g is finitely integrable $\int_\Omega g d\mu < \infty$ under the measure μ.

Lemma 2 is helpful to establishing all three steps of the DCT in the framework here. For illustration, define a measure μ by assigning mass $f(d_1)f(d_2)$ to integer points $(d_1, d_2) \in \Omega \equiv \{(x, y) : \max\{1, \ell\} \leq x \leq y\}$ and define functions as $h_n(d_1, d_2) = \sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s)[1 - p(d_1, d_2, s)]$ on integer pairs

$(d_1, d_2) \in \Omega$, then the integrals are weighted sums

$$\int_\Omega h_n d\mu \equiv \sum_{d_1=\max\{1,\ell\}}^{\infty} \sum_{d_2=d_1}^{\infty} f(d_1) f_2(d_2) h_n(d_1, d_2) = \Delta_{1,n},$$

corresponding to the first quantity in Lemma 1. (However, in Sections 3.4-3.5, a slightly different definition of functions is used to create integrals.)

3.3 Lemma 2

Lemma 2 *Fix integers $d_1, d_2, d_3 \geq \ell$. For an integer $s \geq 0$, let $s^* = s$ if d_1+d_2+s is even and $s^* = s+1$ otherwise (if $d_1+d_2+d_3+s$ is even for (iii) and (iv)). Let C denote a generic constant, not depending on n, d_1, d_2, d_3, s.*
(i) There exists constant $C > 0$ such that, for any $1 \leq d_1 \leq d_2$ and $1 \leq s$,

$$\sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s) \Big| s[1 - p_1(d_1, d_2, s)] \Big| \leq C d_1 d_2.$$

(ii) For any fixed values of $1 \leq d_1 \leq d_2$ and any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{s \geq \delta n} \Big| s[1 - p_1(d_1, d_2, s)] - d_1 d_2 \Big| = 0.$$

(iii) There exists constant $C > 0$ such that, for any $1 \leq d_1 \leq d_2, d_3$ and $n \geq 1$,

$$n^2 \sum_{s=(n-3)\ell}^{\infty} f_{S_{n-3}}(s) p_2(d_1, d_2, d_3, s) \leq C(d_1 d_2 d_3)^{4/3}.$$

(iv) For any fixed values of $1 \leq d_1 \leq d_2, d_3$ and any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{s \geq n\delta} \Big| s^2 p_2(d_1, d_2, d_3, s) - d_1 d_2 (d_1 - 1) d_3 \Big| = 0.$$

Note: It suffices to show the above results for s by the pointwise convergence of $\left(\frac{s+1}{s}\right) \to 1$ and $\left(\frac{s+1}{s}\right)^2 \to 1$.

3.4 Proof of (6) for Theorem 1(i)

For notational convenience, given a function $h(d_1, d_2)$ of integers d_1, d_2, define $\Sigma_{d_1,d_2} h(d_1, d_2) \equiv \sum_{d_1=\max\{\ell,1\}}^{\infty} \sum_{d_2=d_1}^{\infty} f(d_1) f(d_2) h(d_1, d_2)$. Next write

$$(n-1)\Delta_{1,n} = \sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s) \frac{n-1}{s} \cdot \Sigma_{d_1,d_2} d_1 d_2 + \Sigma_{d_1,d_2} r_n(d_1, d_2),$$

where $r_n(d_1,d_2) \equiv \sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s)(n-1)s^{-1}[s(1-p_1(d_1,d_2,s))-d_1d_2]$. Fix $\ell \le d_1 \le d_2$. Then $|(n-1)/S_{n-2}| \le \ell^{-1}$ w.p.1 for all n and $|r_n(d_1,d_2)| \le \sum_{s=(n-1)\ell}^{\infty} f_{S_{n-2}}(s)\ell^{-1}d_1d_2C = \ell^{-1}Cd_1d_2$ holds for all n by Lemma 2(i) (where C does not depend on d_1,d_2). Using the result of Lemma 2(ii), $|r_n(d_1,d_2)| \le \ell^{-1}\sup_{s\ge n\ell/2}|s(1-p_1(d_1,d_2,s))-d_1d_2| \to 0$. By the moment assumption $\mathrm{E}D_1 < \infty$, it holds that $\Sigma_{d_1,d_2}\ell^{-1}Cd_1d_2 = C\ell^{-1}\Sigma_{d_1,d_2}d_1d_2 < \infty$ since

$$\Sigma_{d_1,d_2}d_1d_2 = \frac{1}{2}\left((\mathrm{E}D_1)^2 + \sum_{d=\ell}^{\infty}[d\ f(d)]^2\right) \le (\mathrm{E}D_1)^2 < \infty.$$

Hence, by the DCT, $\lim_{n\to\infty}\Sigma_{d_1,d_2}r_n(d_1,d_2) = \Sigma_{d_1,d_2}\lim_{n\to\infty}r_n(d_1,d_2) = 0$.

By $\mathrm{E}D_1 < \infty$ and the strong law of large numbers (SLLN), $S_{n-2}/(n-1) \to \mathrm{E}D_1 > 0$ with probability 1 (w.p.1) as $n \to \infty$, implying $(n-1)/S_{n-2} \to 1/\mathrm{E}D_1$ w.p.1. The DCT then gives

$$\lim_{n\to\infty}\sum_{s=(n-2)\ell}^{\infty} f_{S_{n-2}}(s)\frac{n-1}{s} = \lim_{n\to\infty}\mathrm{E}\left[\frac{n-1}{S_{n-2}}\right] = \mathrm{E}\left[\lim_{n\to\infty}\frac{n-1}{S_{n-2}}\right] = \mathrm{E}\left[\frac{1}{\mathrm{E}D_1}\right] = \frac{1}{\mathrm{E}D_1}.$$

This establishes (6).

3.5 Proof of (7) for Theorem 1(ii)

Arguments for handling $(n-2)(n-1)/2\cdot\Delta_{2,n}$ are similar to those of the last section. Writing $\Sigma_{d_1,d_2,d_3}h(d_1,d_2,d_3) \equiv \sum_{d_1=\max\{2,\ell\}}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty} f(d_1)f(d_2)f(d_3)h(d_1,d_2,d_3)$ for a given function $h(d_1,d_2,d_3)$, analogously expand

$$\frac{(n-2)(n-1)}{2}\Delta_{2,n} = \frac{1}{2}\mathrm{E}\left[\frac{(n-2)(n-1)}{S_{n-3}^2}\right]\cdot\Sigma_{d_1,d_2,d_3}d_1d_2(d_1-1)d_3 + \Sigma_{d_1,d_2,d_3}r_n(d_1,d_2,d_3)$$

where $\Sigma_{d_1,d_2,d_3}r_n(d_1,d_2,d_3)$ is a remainder defined by substraction. For each fixed $1 \le d_1 \le d_2, d_3$, it holds that

$$\begin{aligned}|r_n(d_1,d_2,d_3)| &\le \sup_{s\ge n\ell/2}|s^2p_2(d_1,d_2,d_3,s)-d_1d_2(d_1-1)d_3|\cdot\frac{1}{2}\sum_{s=(n-3)\ell}^{\infty} f_{S_{n-3}}(s)\left[\frac{(n-2)(n-1)}{s^2}\right] \\ &\le \ell^{-2}\sup_{s\ge n\ell/2}|s^2p_2(d_1,d_2,d_3,s)-d_1d_2(d_1-1)d_3|\end{aligned}$$

so that by, Lemma 2(iv), $\lim_{n\to\infty}r_n(d_1,d_2,d_3) = 0$ with d_1,d_2,d_3 fixed. By Lemma 2(iii), there exists a constant C (not depending on n,d_1,d_2,d_3)

where

$$
\begin{aligned}
|r_n(d_1, d_2, d_3)| &\leq n^2 \sum_{s=(n-3)\ell}^{\infty} f_{S_{n-3}}(s) p_2(d_1, d_2, d_3, s) + \mathrm{E}\left[\frac{(n-1)(n-2)}{S_{n-3}^2}\right] d_1^2 d_2 d_3 \\
&\leq C(d_1 d_2 d_3)^{4/3}
\end{aligned}
$$

(since $d_1 \leq d_2, d_3$ and $\mathrm{E}[(n-1)(n-2)S_{n-3}^{-2}] \leq \ell^{-2}$). The moment assumption $\mathrm{E}D_1^{4/3} < \infty$ implies

$$\Sigma_{d_1,d_2,d_3}(d_1 d_2 d_3)^{4/3} \leq [\mathrm{E}D_1^{4/3}]^3 < \infty.$$

Therefore, the DCT gives $\lim_{n\to\infty} \Sigma_{d_1,d_2,d_3} r_n(d_1, d_2, d_3) = \Sigma_{d_1,d_2,d_3} \lim_{n\to\infty} r_n(d_1, d_2, d_3) = 0$. Using $\mathrm{E}D_1 < \infty$ and SLLN, $(n-1)(n-2)S_{n-3}^{-2} \to (\mathrm{E}D_1)^{-2} > 0$ holds w.p.1 as $n \to \infty$; since $(n-1)(n-2)S_{n-3}^{-2} \leq \ell^{-2}$ for all n, the DCT again yields $\mathrm{E}[(n-1)(n-2)S_{n-3}^{-2}] \to \mathrm{E}[(\mathrm{E}D_1)^{-2}] = (\mathrm{E}D_1)^{-2}$. Hence, (7) follows.

Remark: Many authors have modeled the probability of an edge connecting two nodes as proportional to the product of their degrees. The results in Theorem 1(i) and Theorem 1(ii) are consistent with these authors models as

$$\mathrm{P}(B_n(2)|d_1, d_2) \approx \frac{d_1 d_2}{n\mathrm{E}D_1}$$

and

$$\mathrm{P}(B_n(2), B_n(3)|d_1, d_2, d_3) \approx \frac{d_1 d_2}{n\mathrm{E}D_1}\frac{(d_1-1)d_2}{n\mathrm{E}D_1}.$$

Chung and Lu (2002) define the proportionality constant to be $\rho^{-1} = \sum_{i=1}^n d_i$ making the probability nodes D_1 and D_2 share an edge, $p_{1,2}$, is equal to $d_1 d_2 \rho$. This is equivalent in the limit as

$$p_{1,2} = \frac{d_1 d_2}{\sum_{i=1}^n d_i} \equiv \frac{d_1 d_2}{\frac{n\sum_{i=1}^n d_i}{n}} \overset{SLLN}{\approx} \frac{d_1 d_2}{n\mathrm{E}D_1}.$$

Other authors such as Newman (2003) and Molloy and Reed (1995) have also used the probability proportional to the degrees in their network models.

4 Proof of Lemma 1

4.1 Derivation of $\Delta_{1,n}$

Under the afore mentioned graph generating mechanism, $\Delta_{1,n}$ is derived by considering two cases, $D_1 + D_2 + S$ is even and $D_1 + D_2 + S$ is odd. In the event that the sum is odd, the degree of a randomly selected node in the graph is incremented by one, where all nodes are equally likely to be incremented. Say D_1, D_2, S_{n-2} are the underlying degree random variables and $D_1^\circ, D_2^\circ, S_{n-2}^\circ$ are the "observed" degree random variables. Define a function giving transitional probabilities

$$g(d_1^\circ, d_2^\circ, s^\circ | d_1, d_2, s) = \mathrm{P}(D_1^\circ = d_1^\circ,\ D_2^\circ = d_2^\circ,\ S_{n-2}^\circ = s^\circ\ | D_1 = d_1,\ D_2 = d_2,\ S_{n-2} = s)$$

for integer $d_1^\circ, d_2^\circ, s^\circ, d_1, d_2, s \geq \ell$. This function breaks down into four cases depending on the sum of the degrees.

$$g(d_1^\circ, d_2^\circ, s^\circ | d_1, d_2, s) = \begin{cases} 1 & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ,\ s = s^\circ & d_1 + d_2 + s \text{ even} \\ \frac{n-2}{n} & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ,\ s = s^\circ + 1 & d_1 + d_2 + s \text{ odd} \\ \frac{1}{n} & \text{if } d_1 = d_1^\circ + 1,\ d_2 = d_2^\circ,\ s = s^\circ & d_1 + d_2 + s \text{ odd} \\ \frac{1}{n} & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ + 1,\ s = s^\circ & d_1 + d_2 + s \text{ odd} \end{cases}. \tag{11}$$

As before, let $p(d_1^\circ, d_2^\circ, s^\circ) \equiv$ the probability that nodes 1 and 2 are neighbors given $D_1^\circ = d_1^\circ,\ D_2^\circ = d_2^\circ,\ S_{n-2}^\circ = s^\circ$. Let $B_n(2) \equiv$ "Nodes 1 and 2 are neighbors" and $X_{2,1} \equiv D_1^\circ \leq D_2^\circ$. Define a conditional probability function

$$\begin{aligned} k(d_1^\circ, d_2^\circ, s^\circ) &= \mathrm{P}(B_n(2), X_{2,1} | D_1^\circ = d_1^\circ,\ D_2^\circ = d_2^\circ,\ S_{n-2}^\circ = s^\circ) \\ &= \begin{cases} 0 & \text{if } d_1^\circ > d_2^\circ \\ p_1(d_1^\circ, d_2^\circ, s^\circ) & \text{if } d_2^\circ \geq d_1^\circ \end{cases}. \end{aligned}$$

If $D_1 + D_2 + S_{n-2}$ is even then $D_1 = D_1^\circ,\ D_2 = D_2^\circ, S_{n-2} = S_{n-2}^\circ$; if $D_1 + D_2 + S_{n-2}$ is odd then there are three possibilities: $D_1 = D_1^\circ + 1,\ D_2 = D_2^\circ, S_{n-2} = S_{n-2}^\circ$; $D_1 = D_1^\circ,\ D_2 = D_2^\circ + 1, S_{n-2} = S_{n-2}^\circ$; or $D_1 = D_1^\circ,\ D_2 = D_2^\circ, S_{n-2} = S_{n-2}^\circ + 1$. Decomposing $\mathrm{P}(B_n(2), X_{2,1})$ with the law of total probability gives,

$$\mathrm{P}(B_n(2), X_{2,1}) = \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} \mathrm{P}(D_1 = d_1, D_2 = d_2, S_{n-2} = s, A, B)$$

where $\mathrm{P}(D_1 = d_1, D_2 = d_2, S_{n-2} = s, B_n(2), X_{2,1})$ may be written as

$$f(d_1)f(d_2)f(s) \sum_{\substack{\nu_1,\nu_2,\nu_3 \in \{0,1\} \\ \nu_1+\nu_2+\nu_3 \leq 1}} g(d_1 + \nu_1, d_2 + \nu_2, s + \nu_3 | d_1, d_2, s) k(d_1 + \nu_1, d_2 + \nu_2, s + \nu_3)$$

$$= f(d_1)f(d_2)f(s) \Bigg[p_1(d_1, d_2, s)\mathbb{I}(d_1 + d_2 + s \text{ even}) +$$

$$\mathbb{I}(d_1 + d_2 + s \text{ odd}) \left[\left(\frac{n-2}{n}\right) p_1(d_1, d_2, s+1) + \left(\frac{1}{n}\right) p_1(d_1 + 1, d_2, s) + \left(\frac{1}{n}\right) p_1(d_1, d_2 + 1, s) \right] \Bigg].$$

Hence $\mathrm{P}(B_n(2), X_{2,1}) = I_e + I_{o1} + I_{o2} + I_{o3}$ where

$$I_e = \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2,s)\mathbb{I}(d_1+d_2+s \text{ even})$$

$$\begin{aligned}
I_{o1} &\equiv \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2,s+1)\mathbb{I}(d_1+d_2+s \text{ odd}) \\
&\quad -\frac{2}{n} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2,s+1)\mathbb{I}(d_1+d_2+s \text{ odd}) \\
&\equiv I_{o1}^{(1)} - \frac{2}{n} I_{o1}^{(2)}
\end{aligned}$$

$$I_{o2} \equiv \frac{1}{n} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1+1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1+1,d_2,s)\mathbb{I}(d_1+d_2+s \text{ odd})$$

$$\begin{aligned}
I_{o3} &\equiv \frac{1}{n} \sum_{d_1=\ell}^{\infty} \sum_{\substack{d_2= \\ \max\{d_1-1,\ell\}}}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2+1,s)\mathbb{I}(d_1+d_2+s \text{ odd}) \\
&\equiv \frac{1}{n} \sum_{d_1=\ell+1}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2+1,s)\mathbb{I}(d_1+d_2+s \text{ odd}) \\
&\quad +\frac{1}{n} \sum_{d_1=\ell+1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_1-1)f(s)p_1(d_1,d_1,s)\mathbb{I}(2d_1+s \text{ odd}) \\
&\quad +\frac{1}{n} \sum_{s=(n-2)\ell}^{\infty} f(\ell)f(\ell)f(s)p_1(\ell,\ell,s)\mathbb{I}(2\ell+s \text{ odd}).
\end{aligned}$$

Therefore,

$$\Delta_{1,n} = I_e + I_{o1}^{(1)} = \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{s=(n-2)\ell}^{\infty} f(d_1)f(d_2)f(s)p_1(d_1,d_2,s)$$

where the remainder $R = \mathrm{P}(B_n(2), X_{2,1}) - \Delta_{1,n}$ satisfies

$$\begin{aligned}
|R| &\leq |I_{o1}^{(2)}| + |I_{o2}| + |I_{o3}| \\
&\leq \frac{2}{n^2}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(d_2)\left(n\sum_{s=(n-2)\ell}^{\infty} f(s)\big[p_1(d_1,d_2,s)+p_1(d_1+1,d_2,s)+p_1(d_1,d_2+1,s)\big]\right) \\
&\quad +\frac{1}{n^2}\sum_{d_1=\ell}^{\infty} f(d_1)f(\max\{\ell,d_1-1\})\left(n\sum_{s=(n-2)\ell}^{\infty} f(s)p(d_1,d_1,s)\right) \\
&\leq \frac{C}{n^2}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(d_2)(d_1+1)(d_2+1) + \frac{C}{n^2}\sum_{d_1=\ell}^{\infty} f(d_1)f(\max\{\ell,d_1-1\})d_1^2.
\end{aligned}$$

Using Lemma 2(i) and $\frac{n}{s} \leq 3/\ell$ for $s \geq (n-2)\ell, n \geq 3$. Now

$$\sum_{d1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(d_2)(d_1+1)(d_2+1) \leq \sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(d_2)d_1d_2 \leq 4(ED_1)^2 < \infty$$

and

$$\sum_{d_1=\ell}^{\infty} f(d_1)f(\max\{\ell,d_1-1\})d_1^2 \leq (ED_1)(ED_1+1) < \infty$$

Hence,

$$\mathrm{P}(B_n(2), D_2 \geq D_1) = \Delta_{1,n} + O\left(\frac{1}{n^2}\right).$$

4.2 Derivation of $\Delta_{2,n}$

Similarly to $\Delta_{1,n}$, $\Delta_{2,n}$ requires considering the two two cases, $D_1 + D_2 + D_3 + S$ even and $D_1 + D_2 + D_3 + S$ odd. In the event that the sum is odd, a node in the graph will again be randomly selected to have its degree incremented by one. Considering all nodes equally likely to be incremented, let $D_1+D_2+D_3+S$ be the degree random variables and let $D_1^\circ+D_2^\circ+D_3^\circ+S^\circ$ be the "observed" degree random variables. Define a function

$$\begin{aligned}
&g(d_1^\circ, d_2^\circ, d_3^\circ, s^\circ | d_1, d_2, d_3, s) = \\
&\qquad \mathrm{P}(D_1^\circ = d_1^\circ,\ D_2^\circ = d_2^\circ,\ D_3^\circ = d_3^\circ,\ S_{n-3}^\circ = s^\circ\ | D_1 = d_1,\ D_2 = d_2,\ D_3 = d_3,\ S_{n-3} = s)
\end{aligned} \tag{12}$$

for integers $d_1, d_2, d_3, s, d_1^\circ, d_2^\circ, d_3^\circ, s^\circ \geq \ell$. This function breaks down into five cases depending on the sum of degrees.

$$g(d_1^\circ, d_2^\circ, d_3^\circ, s^\circ | d_1, d_2, d_3, s) =$$

$$\begin{cases} 1 & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ,\ d_3 = d_3^\circ,\ s = s^\circ & d_1 + d_2 + d_3 + s \text{ even} \\ \frac{n-3}{n} & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ,\ d_3 = d_3^\circ,\ s = s^\circ + 1 & d_1 + d_2 + d_3 + s \text{ odd} \\ \frac{1}{n} & \text{if } d_1 = d_1^\circ + 1,\ d_2 = d_2^\circ,\ d_3 = d_3^\circ,\ s = s^\circ & d_1 + d_2 + d_3 + s \text{ odd} \\ \frac{1}{n} & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ + 1,\ d_3 = d_3^\circ,\ s = s^\circ & d_1 + d_2 + d_3 + s \text{ odd} \\ \frac{1}{n} & \text{if } d_1 = d_1^\circ,\ d_2 = d_2^\circ,\ d_3 = d_3^\circ + 1,\ s = s^\circ & d_1 + d_2 + d_3 + s \text{ odd} \end{cases}$$

As before let $h(d_1, d_2, d_3, s) \equiv$ the probability the nodes 1 and 2 are neighbors and nodes 1 and 3 are neighbors given $D_1^\circ = d_1^\circ, D_2^\circ = d_2^\circ, D_3^\circ = d_3^\circ, S^\circ = s^\circ$. Let $B_n(2) \equiv$ "Nodes 1 and 2 are neighbors" and $D_1^\circ \leq D_2^\circ$. Let $B_n(3) \equiv$ "Nodes 1 and 3 are neighbors" and $D_1^\circ \leq D_3^\circ$. Define a function

$$\begin{aligned} k(d_1^\circ, d_2^\circ, d_3^\circ, s^\circ) &= \mathrm{P}(B_n(2), B_n(3)\ \big| D_1^\circ = d_1^\circ, D_2^\circ = d_2^\circ, D_3^\circ = d_3^\circ, S_{n-3}^\circ = s^\circ) \\ &= \begin{cases} 0 & \text{if } d_1^\circ > d_2^\circ \\ 0 & \text{if } d_1^\circ > d_3^\circ \\ p_2(d_1^\circ, d_2^\circ, d_3^\circ, s^\circ) & \text{if } d_2^\circ, d_3^\circ \geq d_1^\circ \end{cases} \end{aligned}$$

Decomposing $\mathrm{P}(B_n(2), B_n(3))$ with the law of total probability it follows,

$$\begin{aligned} \mathrm{P}(B_n(2), B_n(3), X_{2,1}, X_{3,1}) &= \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} \\ &\qquad \mathrm{P}(B_n(2), B_n(3), D_1 = d_1, D_2 = d_2, D_3 = d_3, S_{n-3} = s) \\ &\equiv I_e + I_o \end{aligned}$$

where $\mathrm{P}(B_n(2), B_n(3), D_1 = d_1, D_2 = d_2, D_3 = d_3, S_{n-3} = s)$ may be written as

$$f(d_1)f(d_2)f(d_3)f(s) \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4 \in \{0,1\} \\ \nu_1+\nu_2+\nu_3+\nu_4 \leq 1}} g(d_1+\nu_1, d_2+\nu_2, d_3+\nu_3, s+\nu_4 | d_1, d_2, d_3, s)$$
$$\times\, k(d_1+\nu_1, d_2+\nu_2, d_3+\nu_3, s+\nu_4)$$

$$\begin{aligned} &= f(d_1)f(d_2)f(d_3)f(s) \Bigg[p_2(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + s \text{ even}) + \\ &\quad \mathbb{I}(d_1 + d_2 + s \text{ odd}) \bigg[\left(\frac{n-3}{n}\right) p_2(d_1, d_2, d_3, s+1) + \left(\frac{1}{n}\right) p(d_1+1, d_2, d_3, s) \\ &\qquad + \left(\frac{1}{n}\right) p_2(d_1, d_2+1, d_3, s) + \left(\frac{1}{n}\right) p_2(d_1, d_2, d_3+2, s) \bigg] \Bigg]. \end{aligned}$$

Hence, $\mathrm{P}(B_n(2), B_n(3), X_{2,1}, X_{3,1}) = I_e + I_{o1} + I_{o2} + I_{o3} + I_{o4}$ where

$$\begin{aligned} I_e = & \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)g(d_1, d_2, d_3, s|d_1, d_2, d_3, s) \\ & \qquad\qquad \times k(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ even}) \\ = & \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)h(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ even}) \end{aligned}$$

$$\begin{aligned} I_{o1} \equiv & \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)\frac{n-3}{n}h(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd}) \\ \equiv & \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)h(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd}) \\ & - \frac{3}{n} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)h(d_1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd}) \\ \equiv & I_{o,1}^{(1)} - \frac{3}{n} I_{o1}^{(2)} \end{aligned}$$

$$I_{o2} \equiv \frac{1}{n} \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1+1}^{\infty} \sum_{d_3=d_1+1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)h(d_1+1, d_2, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd})$$

$$\begin{aligned} I_{o3} \equiv & \frac{1}{n} \sum_{d_1=\ell}^{\infty} \sum_{\substack{d_2= \\ \max\{\ell, d_1-1\}}}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)p_2(d_1, d_2+1, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd}) \\ \equiv & \frac{1}{n} \sum_{d_1=\ell+1}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)p_2(d_1, d_1, d_3, s)\mathbb{I}(d_1 + d_2 + d_3 + s \text{ odd}) \\ & + \frac{1}{n} \sum_{d_1=\ell+1}^{\infty} \sum_{d_3=d_1}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_1-1)f(d_3)f(s)p_2(d_1, d_1, d_3, s)\mathbb{I}(2d_1 + d_3 + s \text{ odd}) \\ & + \frac{1}{n} \sum_{d_3=\ell}^{\infty} \sum_{s=(n-3)\ell}^{\infty} f(\ell)f(\ell)f(d_3)f(s)p_2(\ell, \ell, d_3, s)\mathbb{I}(2\ell + d_3 + s \text{ odd}) \end{aligned}$$

$$
\begin{aligned}
I_{o4} \equiv &\frac{1}{n}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{\substack{d_3=\\ \max\{\ell,d_1-1\}}}^{\infty}\sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)p_2(d_1,d_2+1,d_3,s)\mathbb{I}(d_1+d_2+d_3+s \text{ odd})\\
\equiv &\frac{1}{n}\sum_{d_1=\ell+1}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty}\sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)p_2(d_1,d_3,d_1,s)\mathbb{I}(d_1+d_2+d_3+s \text{ odd})\\
&+\frac{1}{n}\sum_{d_1=\ell+1}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_1-1)f(s)p_2(d_1,d_2,d_1,s)\mathbb{I}(2d_1+d_2+s \text{ odd})\\
&+\frac{1}{n}\sum_{d_2=\ell}^{\infty}\sum_{s=(n-3)\ell}^{\infty} f(\ell)f(d_2)f(\ell)f(s)p_2(\ell,d_2,\ell,s)\mathbb{I}(2\ell+d_2+s \text{ odd}).
\end{aligned}
$$

Consequently,

$$
\Delta_{2,n} = I_e + I_{o1}^{(1)} = \sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty}\sum_{s=(n-3)\ell}^{\infty} f(d_1)f(d_2)f(d_3)f(s)p_2(d_1,d_2,d_3,s)
$$

where the remainder $R = \mathrm{P}(B_n(2), B_n(3), X_{2,1}, X_{3,1}) - \Delta_{2,n}$ satisfies

$$
\begin{aligned}
R \leq\ & |I_{o1}^{(2)}| + |I_{o2}| + |I_{o3}| + |I_{o4}|\\
\leq\ & \frac{3}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty} f(d_1)f(d_2)f(d_3)\Bigg(n^2\sum_{s=(n-3)\ell}^{\infty} f(s)\big[p_2(d_1+1,d_2,d_3,s)+p_2(d_1,d_2+1,d_3,s)\\
&+p_2(d_1,d_2,d_3+1,s)+p_2(d_1,d_2,d_3,s+1)\big]\Bigg)\\
&+\frac{1}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_2)\left(n^2\sum_{s=(n-3)\ell}^{\infty} f(s)p_2(d_1,d_2,d_1,s)\right)\\
&+\frac{1}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_3=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_3)\left(n^2\sum_{s=(n-3)\ell}^{\infty} f(s)p_2(d_1,d_1,d_3,s)\right)\\
\leq\ & \frac{C}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty}\sum_{d_3=d_1}^{\infty} f(d_1)f(d_2)f(d_3)(d_1+1)(d_2+1)(d_1)(d_3+1)\\
&+\frac{C}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_2=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_2)d_1^2(d_1-1)d_3\\
&+\frac{C}{n^3}\sum_{d_1=\ell}^{\infty}\sum_{d_3=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_3)d_1d_2(d_1-1)d_1.
\end{aligned}
$$

Using Lemma 2(iii), $\frac{n^2}{s^2} \leq 1$ and $s \geq (n-3)\ell, n \geq 4$,

$$\begin{aligned}
&\sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} f(d_1)f(d_2)f(d_3)(d_1+1)(d_2+1)(d_1)(d_3+1) \\
&\leq \sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} \sum_{d_3=d_1}^{\infty} f(d_1)f(d_2)f(d_3)4d_1d_2(d_1-1)d_3 \\
&\leq 4\mathrm{E}(D_1D_2D_3)^{4/3} < \infty
\end{aligned}$$

and

$$\sum_{d_1=\ell}^{\infty} \sum_{d_2=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_2)d_1^2(d_1-1)d_3 \leq \mathrm{E}(D_1D_2D_3)^{4/3} < \infty$$

and

$$\sum_{d_1=\ell}^{\infty} \sum_{d_3=d_1}^{\infty} f(d_1)f(\max\{d_1-1,\ell\})f(d_3)d_1d_2(d_1-1)d_1 \leq \mathrm{E}(D_1D_2D_3)^{4/3} < \infty.$$

Hence

$$\mathrm{P}(B_n(2), B_n(3), D_3 \geq D_1, D_2 \geq D_1) = \Delta_{2,n} + O\left(n^{-3}\right).$$

5 Proof of Lemma 2

Without loss of generality consider nodes D_1, D_2 and D_3 each having d_1, d_2 and d_3 stubs. To evaluate the limit, each factorial involving s in (8) is approximated using Stirling's Approximation to $\log(s!)$

$$\log(s!) = \frac{\log(\sqrt{2\pi})}{2} + s\log s - s + \frac{1}{12s} + O\Big(\frac{1}{s^3}\Big). \tag{13}$$

This changes the expression of each of the disjoint $h(d_1, d_2, s, k)$ from (8) to

$$\begin{aligned}
h(d_1, d_2, s, k) = {}& \frac{d_1!\, e^k}{(d_1-2k)!k!2^k} \frac{1}{s^k} \Big(1 - \frac{(d_1-2k)}{s}\Big)^{(s-(d_1-2k)+1/2} \Big(1 + \frac{d_1+d_2}{s}\Big)^{(s+d_1+d_2)/2} \\
& \times \Big(1 + \frac{d_2-(d_1-2k)}{s}\Big)^{(s+d_2-(d_1-2k))/2} \times R(d_1, d_2, s, k)
\end{aligned} \tag{14}$$

where

$$R(d_1, d_2, s, k) = \exp\left(\frac{4k-3d_1}{s^2} + d_1d_2\, O\left(s^{-3}\right)\right).$$

5.1 Proof for Lemma 2(i)

Consider $\big|s[1-p_1(d_1,d_2,s)]\big|$. Since $[1-p_1(d_1,d_2,s)] \leq 1$ for all d_1,d_2,s, it follows, $\big|s[1-p_1(d_1,d_2,s)]\big| < 4d_1d_2 \leq Cd_1d_2$ for $s \leq 4d_1d_2$. Therefore, it is left to show for $s > 4d_1d_2$ there exists a constant $C > 0$ such that $s[1-p_1(d_1,d_2,s)] < Cd_1d_2$. To show this requires the decomposition of $s[1-p_1(d_1,d_2,s)]$ into its component parts. Stirling's approximation (13) is then used to establish the existence of the bound. Taylor's Theorem is used to evaluate two cases, no self-links and at least one self link. In the notation this corresponds to $k=0$ and $k>0$. Substitution of Stirling's approximation (13) for each factorial involving s yields,

$$\begin{aligned} g(k,d_1,d_2,s) = {} & \frac{d_1!\, e^k}{(d_1-2k)!k!2^k}\frac{1}{s^k}\Big(1-\frac{(d_1-2k)}{s}\Big)^{(s-(d_1-2k)+1/2}\Big(1+\frac{d_1+d_2}{s}\Big)^{(s+d_1+d_2)/2} \\ & \times\Big(1+\frac{d_2-(d_1-2k)}{s}\Big)^{(s+d_2-(d_1-2k))/2} \times R(k,d_1,d_2,s) \end{aligned} \tag{15}$$

where

$$R(k,d_1,d_2,s) = \exp\left(\frac{4k-3d_1}{s^2} + d_1d_2\,O\left(s^{-3}\right)\right).$$

Using the taylor expansions for $\exp(x)$ and $\log(1+x)$ it follows,

$$g(k,d_1,d_2,s) = \frac{d_1!e^k}{(d_1-2k)!k!2^k}\frac{1}{s^k}\exp\Big\{q(d_1,d_2,s,k)\Big\} \times R(k,d_1,d_2,s)$$

where

$$\begin{aligned} q(d_1,d_2,s,k) = {} & -k + \sum_{r=1}^{\infty}(-1)^r\frac{(2k-d_1)^{(r+1)}}{(r(r+1))s^r} + \sum_{r=1}^{\infty}(-1)^r\frac{(2k-d_1)^r}{2(r)s^r} \\ & + \sum_{r=1}^{\infty}(-1)^{r+1}\left(\frac{(d_2+2k-d_1)^{(r+1)}}{(2r(r+1))s^r} - \frac{(d_2+d_1)^{(r+1)}}{(2r(r+1))s^r}\right). \end{aligned} \tag{16}$$

Fix $1 \leq k \leq \lfloor d_1/2 \rfloor$. A bound for each of the summations from (16) will be shown to exist using $d_1^2 \leq d_1d_2 \leq d_2^2$ and $(d_1+d_2)^m \leq (2d_2)^m$ for $d_1 \leq d_2$ along with $\frac{1}{4d_1d_2} > \frac{1}{4d_1d_2+1}$. When $d_1 \geq 1$ it holds,

$$\left|\sum_{r=1}^{\infty}(-1)^r\frac{(2k-d_1)^{(r+1)}}{(r(r+1))s^r}\right| \leq \sum_{r=1}^{\infty}\frac{(d_1)^{(r+1)}}{(r^2)4^r(d_1)^{2r}} \leq \frac{1}{4}.$$

Similarly,

$$\left|\sum_{r=1}^{\infty}(-1)^r\frac{(2k-d_1)^r}{2(r)s^r}\right| \leq \sum_{r=1}^{\infty}\frac{(d_1)^r}{2(r)4^r(d_1)^{2r}} \leq \frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{4^r} = \frac{2}{3}.$$

The result $|(a-b)^{m+1}-(a+b)^{m+1}| \leq (m+1)(2b)^m(2a)$, for $b \geq a$, $m \geq 1$ along with the triangle inequality establish the bound

$$\begin{aligned}\sum_{r=1}^{\infty}&\left|(-1)^{r+1}\left(\frac{(d_2+2k-d_1)^{(r+1)}}{(2r(r+1))s^r}-\frac{(d_2+d_1)^{(r+1)}}{(2r(r+1))s^r}\right)\right|\\ &\leq \sum_{r=1}^{\infty}\frac{(r+1)(2d_2)^r\,2(d_1-2k)}{2r(r+1)(4d_1d_2)^r}\\ &\leq \sum_{r=1}^{\infty}\frac{1}{2^r}=2.\end{aligned}$$

Since $\max_{1\leq k\leq\lfloor d_1/2\rfloor} R(k,d_1,d_2,s) \leq C$ for a constant $C>0$ not depending on d_1,d_2 where $s \geq 4d_1d_2$, it holds that

$$\sum_{k=1}^{\lfloor\frac{d_1}{2}\rfloor} g(k,d_1,d_2,s) \leq C\sum_{k=1}^{\lfloor\frac{d_1}{2}\rfloor}\frac{d_1!}{(d_1-2k)!}\frac{1}{2^ks^k} \leq C\frac{d_1^2}{s}\sum_{i=1}^{\infty}\frac{1}{2^k} \leq 2C\frac{d_1d_2}{s}.$$

Now consider $k=0$ and $g(k=0,d_1,d_2,s)$ for $s \geq 4d_1d_2$. In this case, $R(0,d_1,d_2,s)=\exp(-d_1/(4s^2)+d_1d_2O(s^{-3}))$ and (16) bound as $|h(d_1,d_2,s,k=0)| \leq 4d_1d_2/s$, using similar arguments as before. Hence, $|s(1-g(k=0,d_1,d_2,s))| \leq Cd_1d_2$ for $s \geq 4d_1d_2$ using that $e^x = 1+xe^a$ for some $0 \leq a \leq x$.

5.2 Proof of Lemma 2(ii)

Fix integers $d_1,d_2 \geq \ell$. Using Stirling's approximation from (13) to evaluate the behavior of s in the permutation functions in $h(d_1,d_2,s,k)$ from (9), it is shown that as a function of $k \geq 0$,

$$\begin{aligned} h(d_1,d_2,s,0) &= \exp\left[-\frac{d_1(d_1-1)}{2s}-\frac{d_1d_2}{s}+O\left(\frac{1}{s^2}\right)\right] \qquad k=0\\ h(d_1,d_2,s,1) &= \frac{d_1(d_1-1)}{2s}\exp\left[O\left(\frac{1}{s}\right)\right] \qquad k=1\\ h(d_1,d_2,s,k) &= O\left(\frac{1}{s^k}\right) \qquad k\geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}\big(1-p_1(d_1,d_2,s)\big) &= 1-\left[h(d_1,d_2,s,0)+h(d_1,d_2,s,1)+O\Big(\frac{1}{s^2}\Big)\right]\\ &= \frac{d_1d_2}{s}+O\Big(\frac{1}{s^2}\Big).\end{aligned}$$

Therefore, as $s\to\infty$, $s\big(1-p_1(d_1,d_2,s)\big)\to d_1d_2$. Hence, for $\delta>0$ and for fixed $d_2\geq d_1$

$$\lim_{n\to\infty}\sup_{s\geq\delta n}\Big|s[1-p_1(d_1,d_2,s)]-d_1d_2\Big|=0.$$

5.3 Proof for Lemma 2(iii)

Consider $\big|n^2[p_2(d_1,d_2,d_3,s)]\big|$ where $p_2(d_1,d_2,d_3,s)=1-p_1(d_1,d_2,s+d_3)-p_1(d_1,d_3,s+d_2)+p_1(d_1,d_2+d_3,s)$. There are three cases to consider: (1) $d_1d_2>n/6$, (2) $d_1d_2\leq n/6$ and $d_1d_3>s/36$ and (3) $d_1d_2\leq n/6$ and $d_1d_3<s/36$. Without loss of generality consider $d_1\leq d_2\leq d_3$. If not then $d_1\leq d_3\leq d_2$ will be the case and the three cases to consider will change only in that d_2 and d_3 will switch places. Substituting Stirling's approximation for each factorial involving s we see

In the case $d_1d_2>n/6$ it holds,

$$n^2\sum_{s=(n-3)\ell}^{\infty}f_{S_{n-3}}p_2(d_1,d_2,d_3,s)\leq 36(d_1d_2)^2\leq 36(d_1d_2d_3)^{4/3}<\infty.$$

In the second case,

$$\begin{aligned}s|p_2(d_1,d_2,d_3,s)| &\leq s\big|1-p_1(d_1,d_2,s+d_3)\big|+s\big|p_1(d_1,d_2+d_3,s)-p_1(d_1,d_3,s+d_2)\big|\\ &\leq Cd_1d_2+s\left|\frac{d_1(d_2+d_3)}{s}-\frac{d_1d_3}{s}+O\left(\frac{1}{s^2}\right)\right|\leq 2Cd_1d_2.\end{aligned}$$

Since $d_1d_3>s/36>n/36$

$$n^2\sum_{s=(n-3)\ell}^{\infty}\frac{f_{S_{n-3}}}{s}s\ p_2(d_1,d_2,d_3,s)\leq Cd_1d_2(36d_1d_3)\leq 36C(d_1d_2d_3)^{4/3}<\infty.$$

Finally there is the case $d_1d_2\leq n/6$ and $d_1d_3<s/36$. One way to approach this situation is by proceeding as in Lemma 2i and considering the cases

$k = 0, 1$ and $k \geq 2$ then using $e^x \leq 1 + x + \frac{x^2}{2} e^a$ for some $a \in [0, x]$. For $k \geq 2$,

$$3 \sum_{k=2}^{\infty} \frac{d_1!}{(d_1 - 2k)! s^{k-2} 2^k} \leq M d_1^4 \sum_{k=2}^{\infty} 12^k \leq \frac{3 M d_1^4}{2} \leq M'(d_1 d_2 d_3)^{4/3} < \infty.$$

Where $M' \in (0, \infty)$ is the bound obtained as in Lemma 2i since $s > 36 d_1 d_3$. For $k = 0$ it follows

$$\begin{aligned}
s^2 |p_2(d_1, d_2, d_3, s)| &= s^2 \big| 1 - p_1(d_1, d_2,\ s + d_3) - p_1(d_1, d_3,\ s + d_2) + p_1(d_1,\ d_2 + d_3,\ s) \big| \\
&\leq s^2 \Bigg| \frac{d_1 d_2}{s + d_3} + e^{Eq.(16)} \frac{1}{2} \left(\frac{d_1 d_2}{s + d_3} \right)^2 + \frac{d_1 d_3}{s + d_2} + e^{Eq.(16)} \frac{1}{2} \left(\frac{d_1 d_3}{s + d_2} \right)^2 \\
&\qquad - \frac{d_1 (d_2 + d_3)}{s} + e^{Eq.(16)} \frac{1}{2} \left(\frac{d_1 (d_2 + d_3)}{s} \right)^2 \Bigg| \\
&\leq s^2 \left| \frac{d_1 d_2}{s + d_3} + \frac{d_1 d_3}{s + d_2} - \frac{d_1 (d_2 + d_3)}{s} \right| \\
&\qquad + s^2 \left| \frac{1}{2} e^{Eq.(16)} \right| \left| \left(\frac{d_1 d_2}{s + d_3} \right)^2 + \left(\frac{d_1 d_3}{s + d_2} \right)^2 - \left(\frac{d_1 (d_2 + d_3)}{s} \right)^2 \right| \\
&\leq C (d_1 d_2 d_3)^{4/3} < \infty.
\end{aligned}$$

For $k = 1$ it holds that $g(k = 2, d_1, d_2, d_3, s) \leq g(k = 1, d_1, d_2, d_3, s) \leq g(k = 0, d_1, d_2, d_3, s) \leq C(d_1 d_2 d_3)^{4/3} < \infty$. Hence, the DCT can be applied allowing the limit and the summation to be interchanged provided $E D_1^{4/3} < \infty$.

5.4 Proof of Lemma 2(iv)

Fix integers $d_1, d_2, d_3 \geq 0$. Again using Stirling's approximation (13) to evaluate the behavior of s in each of the permutation functions in (10), it is shown that for $k \geq 0$,

$$
\begin{aligned}
k \geq 0 \qquad h(d_1, d_2, s, k) &= O\left(\frac{1}{s^k}\right) \\
k = 2 \qquad h(d_1, d_2, s, 2) &= \frac{d_1(d_1-1)(d_1-2)(d_1-3)}{8s^2} + O\Big(\frac{1}{s^3}\Big) \\
k = 1 \qquad h(d_1, d_2, s, 1) &= \frac{d_1(d_1-1)}{2s} \exp\left[\frac{(d_2+1)(1-d_1)}{s} - \frac{1}{2}\frac{(d_1-2)(d_1-3)}{s} + O\Big(\frac{1}{s^2}\Big)\right] \\
k = 0 \qquad h(d_1, d_2, s, 0) &= \exp\Bigg[-\frac{d_1(d_1-1)}{2s} - \frac{d_1 d_2}{s} + \frac{1}{s^2}\bigg(\frac{d_2^2 d_1}{2} \\
&\quad + \frac{d_2^2 d_1^2}{2} + \frac{d_1^2(d_1-1)d_2}{2} + \frac{d_1^2(d_1-1)^2}{8} + \frac{d_1(d_1-1)}{4}\bigg) + O\Big(\frac{1}{s^3}\Big)\Bigg].
\end{aligned}
$$

When $k = 0$ there is a contribution of $\exp\left(\frac{-d_1}{4s^2}\right)$ to this from the $\frac{1}{12s}$ term of (13).

$$
\begin{aligned}
\big(1 - p_1(d_1, d_2, s)\big) &= 1 - \left[h(d_1, d_2, s, 0) + h(d_1, d_2, s, 1) + h(d_1, d_2, s, 2) + O\Big(\frac{1}{s^3}\Big)\right] \\
&= \frac{d_1 d_2}{s} - \frac{1}{s^2}\Bigg[\frac{d_2^2 d_1}{2} + \frac{d_2^2 d_1^2}{2} + \frac{d_1^2(d_1-1)d_2}{2} + \frac{d_1(d_1-1)}{4} + \frac{d_1^2(d_1-1)^2}{8} \\
&\quad - \frac{d_1(d_1-1)^2(d_2+1)}{2} - \frac{d_1(d_1-1)(d_1-2)(d_1-3)}{8}\Bigg] + O\Big(\frac{1}{s^3}\Big).
\end{aligned}
$$

Using the decomposition:

$$
p_2(d_1, d_2, d_3, s) = \big(1 - p_1(d_1, d_2,\ s_2 + d_3)\big) + \big(1 - p_1(d_1, d_3,\ s_2 + d_2)\big) - \big(1 - p_1(d_1,\ d_2 + d_3,\ s_2)\big)
$$

as $s \to \infty$, $s^2\big(p_2(d_1, d_2, d_3, s)\big) \to d_1 d_2 (d_1 - 1) d_3$. Hence, for $\delta > 0$ and for fixed $d_2 \geq d_1$ and $d_3 \geq d_1$

$$
\lim_{n\to\infty} \sup_{s \geq \delta n} \left| s^2 p_2(d_1, d_2, d_3, s) - d_1 d_2 (d_1 - 1) d_3 \right| = 0.
$$

6 Simulations

Simulations from three different degree distributions, each with different moment properties, were performed to examine the asymptotic behavior of the expectations for both the size of a bucket and the expected number of pairs in a bucket. Graphs of size $n = 50, 100, 500, 1000, 5000$ were generated using the configuration model. For each graph size and degree distribution, 4000 random graphs were generated under the configuration model. In the event that the sum of the degrees was odd in a graph generation, one of the nodes was randomly selected and incremented by one.

The expected size $\mathrm{E}N_{i,n}$ of a bucket and the expected number of pairs $\mathrm{E}\binom{N_{i,n}}{2}$ in a bucket were approximated as follows. For a given graph size n, degree distribution and Monte Carlo run $j = 1, \ldots, 4000$, let $N_{i,n}^{(j)}$ and $D_{i,n}^{(j)}$ denote the bucket size and observed degree for node $i = 1, \ldots, n$. Monte Carlo approximations for $\mathrm{E}N_{i,n}$, $\mathrm{E}\binom{N_{i,n}}{2}$, and $\mathrm{E}D_{i,n}^{(j)}$ were computed as

$$\frac{1}{4000}\sum_{j=1}^{4000}\left(\frac{1}{n}\sum_{i=1}^{n} N_{i,n}^{(j)}\right), \quad \frac{1}{4000}\sum_{j=1}^{4000}\left(\frac{1}{n}\sum_{i=1}^{n}\binom{N_{i,n}^{(j)}}{2}\right), \quad \frac{1}{4000}\sum_{j=1}^{4000}\left(\frac{1}{n}\sum_{i=1}^{n} D_{i,n}^{(j)}\right),$$

which correspond to Monte Carlo averages of within-graph sample averages. These approximates are reported in Tables 1-3 to follow, separated by degree distribution. These tables also include a value "∞" which denotes the theoretical limits of $\mathrm{E}N_{i,n}$ and $\mathrm{E}\binom{N_{i,n}}{2}$ from Theorem 1 as well as the limit $\lim_{n\to\infty}\mathrm{E}D_{i,n} = \mathrm{E}D_1$ (where $\mathrm{E}D_{i,n}$ may differ from $\mathrm{E}D_1$ due to imcrementing a random node by 1 in the configuration model). For each graph size n and degree distribution, $\frac{1}{4000}\sum_{j=1}^{4000}\left(\sum_{i=1}^{n}\binom{N_{i,n}^{(j)}}{2}\right)$ was computed as the Monte Carlo approximation of $n\mathrm{E}\binom{N_{i,n}}{2} = \sum_{i=1}^{n}\mathrm{E}\binom{N_{i,n}}{2}$, the expected number of bucket pairs summed over all nodes in the graph. These values are displayed in Figures 1-3 to follow. **R** code for both the simulations and the theoretical limits can be seen in the appendix sections "Empirical Results" and "Theoretical Results", respectively.

Two power law degree distributions were manually programmed for the simulations. The first was a power law with exponent $\beta = 2.4$ and the second was a power law with exponent $\beta = 3.2$. To simplify the simulations, both power law distributions were truncated at 50,000. Using a continuous approximation for both power law distributions used shows that the probability of observing a degree greater than 50,000 is approximately

$$\mathrm{P}(D_i > 50,000|\beta) \;=\; \sum_{d=50,001}^{\infty} d^{-\beta} \;<\; \int_{50,000}^{\infty} (\beta-1)x^{-\beta}\,dx \;=\; 50,000^{1-\beta}.$$

For $\beta = 2.4$, $\mathrm{P}(D_i > 50,000) < 2.63 \times 10^{-7}$. For $\beta = 3.2$, $\mathrm{P}(D_i > 50,000) < 4.59 \times 10^{-11}$. Consequently, this is a conservative truncation point that will not affect graphs of the size generated. The third distribution was a Poisson distribution with parameter $\lambda = 4.0$ which was truncated at 500 due to numerical restrictions. That is, the probability of observing a degree greater than 500 was less than 1×10^{-350}. See the "Distributions" section of the appendix for **R** code.

6.1 Power Law $\beta = 2.4$

A power law degree distribution with exponent $\beta = 2.4$ was chosen as it has a finite 4/3 moment but does not have a finite second moment. As can be seen in Table 1, the convergence of the expected bucket size and the expected number of pairs in a bucket to the theoretical limits from Theorem 1 is slow. Figure 1 plots the expected total number of bucket pairs over all graph nodes

n	$\mathrm{E}N_{i,n}$	$\mathrm{E}\binom{N_{i,n}}{2}$	$\mathrm{E}D_{i,n}$
50	1.029	0.103	2.219
100	1.068	0.135	2.291
500	1.138	0.217	2.223
1000	1.161	0.256	2.215
5000	1.202	0.345	2.233
∞	1.257	0.687	2.221

Table 1: Power law degree distribution with $\alpha = 2.4$.

against the size of a graph. The figure also displays $n \cdot \lim_{n\to\infty} \mathrm{E}\binom{N_{i,n}}{2}$ (graph size times theoretical limit) plotted as a straight line function of graph size n. Because of the slow convergence, the empirical expectation also depends on the size of the graph. The solid line shows that the expected number of pairs in a bucket grows linearly with the number of nodes in a graph.

6.2 Power Law $\beta = 3.2$

A power law degree distribution with exponent $\beta = 3.2$ was chosen because it has a finite second moment. As can be seen in Table 1, the convergence

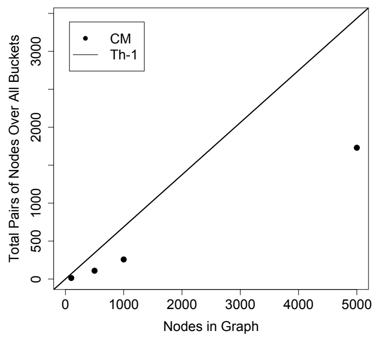

Figure 1: Scales at most linearly when the second moment is undefined.

of the expected bucket size and the expected number of pairs in a bucket to the theoretical limits from Theorem 1 is quite reasonable. A graph with 500 nodes already has two significant digits for the expected size of a bucket

n	$\mathrm{E}N_{i,n}$	$\mathrm{E}\binom{N_{i,n}}{2}$	$\mathrm{E}D_{i,n}$
50	0.931	0.0123	1.283
100	0.935	0.0130	1.281
500	0.941	0.0142	1.280
1000	0.942	0.0142	1.277
5000	0.943	0.0145	1.278
∞	0.943	0.0146	1.277

Table 2: Power law degree distribution with $\alpha = 3.2$.

and three significant digits for the expected number of pairs in a bucket. The linear increase in the expected number of pairs in a bucket with the size of the graph can clearly be seen in Figure 2. This suggests that the convergence of $\mathrm{E}N_{i,n}$ and $\mathrm{E}\binom{N_{i,n}}{2}$ to their theoretical limits will be fast for all power law degree distributions with a finite second moment.

6.3 Poisson with $\lambda = 4.0$

A Poisson degree distribution with parameter $\lambda = 4.0$ was chosen as an example of a degree distribution where all the moments are finite. As can be seen in Table 3, the convergence of the expected bucket size and the expected

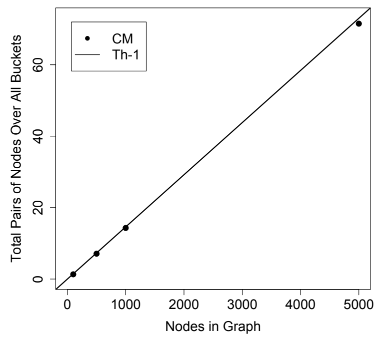

Figure 2: Scales linearly when the second moment is finite.

number of pairs in a bucket to the theoretical limits from Theorem 1 is quite fast. The difference from the $\beta = 3.2$ case is perhaps that $\mathrm{E}\sum_{i=1}^{n}\binom{N_{i,n}}{2}$

n	$\mathrm{E}N_{i,n}$	$\mathrm{E}\binom{N_{i,n}}{2}$	$\mathrm{E}D_{i,n}$
50	2.20	1.90	4.09
100	2.26	2.01	4.08
500	2.31	2.10	4.07
1000	2.32	2.12	4.07
5000	2.328	2.127	4.074
∞	2.329	2.130	4.074

Table 3: Poisson degree distribution with $\lambda = 4.0$.

converges to $n \cdot \lim_{n\to\infty} \mathrm{E}\binom{N_{i,n}}{2}$ a little better at all graph sizes (including $n = 5000$). The linear increase in the expected number of pairs in a bucket as the size of the graph increases can clearly be seen in Figure 3.

7 Conclusion

As the networks being studied continue to grow, the time and space requirements of the algorithms used in complex network analysis becomes a major concern. In order for the "bucket" algorithm to be an effective node counting algorithm, these requirements must grow at most linearly for relevant degree distributions. The empirical evidence suggests that many of the

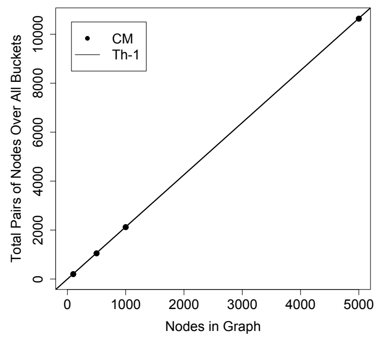

Figure 3: Scales linearly when every moment is finite.

real-world networks be modeled with a power law degree distribution with exponent $2 < \beta < 3$. By showing the expected number of pairs of nodes in a bucket is finite provided the degree distribution has a finite 4/3 moment, a mild condition was given under which the requirements for the "bucket" algorithm grow at most linearly. This is an original and non-trivial result as the lack of a finite second moment for the desired power law distributions is almost always mitigated by truncation. Further our results hold for any arbitrary degree distribution, showing that the "bucket" algorithm will be an effective node counting algorithm for scale-free networks with an exponent $\beta > 2\frac{1}{3}$.

Appendix

Here is the R code used for the simulations.

Distributions

```
##Power Law Beta = 2.4
m<-50000
alpha<-2.4
v<-seq(1,m,1)^(-alpha)
v<-v/sum(v)

##Power Law Beta = 3.2
m<-50000
alpha<-3.2
v<-seq(1,m,1)^(-alpha)
v<-v/sum(v)

##Poisson (lambda = 4.0)
m <- 500  ##computational limits, probabilities were 0 by 500
lambda <- 4.0
v<-dpois(m,lambda)
v<-v/sum(v)
```

Theoretical Limits

```
ED<-sum(v*seq(1,m,1))
EN<-0
EN2<-0
for(i in 1:m){
tmp<-sum(seq(i,m,1)*v[i:m])
EN<-EN + v[i]*i*tmp
EN2<-EN2 + v[i]*i*(i-1)*(tmp^2)
}
EN<-EN/ED
EN2<-EN2/(2*ED^2)
```

Empirical Results

```
# specify M= # of Monte Carlo runs
M<-1000
res2<-matrix(0,ncol=3,nrow=M)

### specify "n" here
n<-50

for(jj in 1:M){

## sample degrees
t<-sample.int(m,n,replace=TRUE,prob=v)

## adjust if sum of degrees is odd
tmp<-sum(t)/2
tmp<- tmp-floor(tmp)

if(tmp>0){
r<- ceiling(runif(1)*n)
t[r]<-t[r]+1
}

### random wiring: t1 & t2 are final vectors, components of t1 are
### matched to components of t2 to form edges between nodes

### the vector "tt" below contains the value "i" exactly D_i times
### for i=1,...,n where D_i is the degree drawn for node i

t1<-rep(seq(1,n,1),t)
tt<-t1
n1<-length(t1)
t3<-sample(seq(1,n1,1))

t1<-tt[t3[1:(n1/2)]]
t2<-tt[t3[(1+n1/2):n1]]

td1<-t1
td2<-t2
```

```
n1<-length(t1)
J<-matrix(0,ncol=n,nrow=n)
J1<-J

for(i in 1:n1){
a1<-t1[i]
a2<-t2[i]

if(a1!=a2){
J1[a1,a2]<-1
J1[a2,a1]<-1
}

J[a1,a2]<-1+J[a1,a2]
J[a2,a1]<-1+J[a2,a1]
}

d<-apply(J,1,sum)
nn<-0*d
nn2<-0*d

for(i in 1:n){
a1<-d[i]
a2<-J1[i,]
a3<-seq(1,n,1)[a2==1]
a3<-d[a3]
nn[i]<-length(a3[a3>=a1])
nn2[i]<- nn[i]*(nn[i]-1)/2
}

res2[jj,]<-c(mean(nn),mean(nn2),mean(d))
print(jj)
}

### for CM, returns approximation to EN_n, EN2_n, ED_n
CM<-apply(res2,2,mean)
```

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